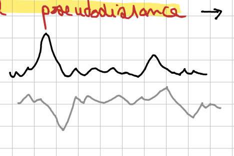
Lectures on TDA   8 JAN

What is the SHAPE of data? → NON TRIVIAL

1- Geometric Simplicial Complexes
2- Homology Groups → Algebraic tools that describe holes in a simplicial complex
3- Natural pseudodistance → Comparing data under the action of a group

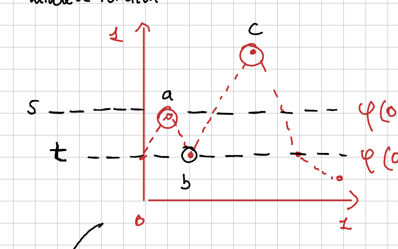


→ HOW MUCH DIFFERENT?

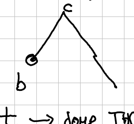
Can I translate the TS?
Can I NOT translate it?

4- Persistence Homology & Persistence Diagrams

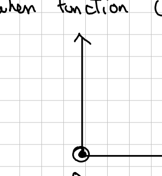
Imagine discrete function $\varphi : [0,1] \to [0,1]$



What's its SHAPE? → Looking at MIN & MAX only we could say



Consider when function $\varphi \ll t$ → some THRESHOLD



2 connected components

When $\varphi \ll s \Rightarrow$ 3 CONNECTED COMPONENTS

What if our threshold can go from $(-\infty, +\infty)$?

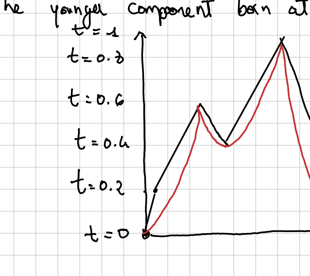
$t = 0 \Rightarrow$ 1 COMPONENT $(0, \infty)$ ↳ component born at 0 never dies

$t = 0.2 \Rightarrow$ 2 COMPONENT $(0.2, 1)$

⋮ $(0.4, 0.6)$

t birth t, death

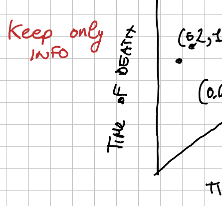
The younger component born at 0.4 dies at 0.6



PERSISTENT DIAGRAM

HAS SAME PERSISTENT DIAGRAM

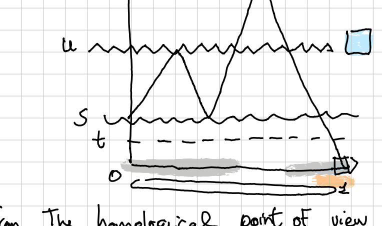
Obviously we keep only more important info



Not easy to extend to MANIFOLDS

Let's try with a continuous function

φ continuous



From the homological point of view □ and ⊂⊃ are the same ↳ n-connected compone

Ex. Natural PSEUDODISTANCE $d_G(\varphi, \psi)$
↳ disturb on temperature today
↳ disturb of temp last year

We now that: STABILITY THEOREM

$O(n \log n) \leftarrow d_G(\varphi, \psi) \geq d_{MATCH}(D_{gm}(\varphi), D_{gm}(\psi))$

GENEOs : group equivariant non-expansive operators
(generalize equivariant NN)
↳ behaves well to minimal changes

They generalize the persistence diagram

Equivariant   Isometries ⊕ Blurring
or
Blurring ⊕ Isometries

Blurring is equivariant
(doesn't change distance of data)

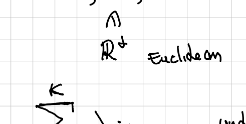
Geometric Simplicial Complexes

Def: Affine Combinations of vectors $u_0, \dots, u_K$ with coefficients $\lambda_0, \dots \lambda_K$

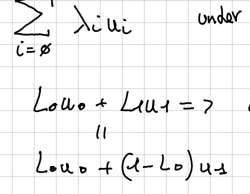
$\mathbb{R}^d$ Euclidean

$$\sum_{i=0}^{K} \lambda_i u_i \quad \text{under the assumption} \quad \sum_{i=0}^{K} \lambda_i = 1$$

So $L_0 u_0 + L_1 u_1 \Rightarrow L_0 + L_1 = 1$
$L_1 = 1 - L_0$
$= L_0 u_0 + (1 - L_0) u_1$ with $L_0 \in \mathbb{R}$



Now



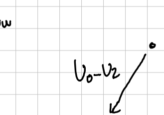
$\lambda_0 u_0 + \lambda_1 u_1 + \lambda_2 u_2$
$\lambda_0 + \lambda_1 + \lambda_2 = 1$

Plane with 3 free points.

The set of any affine combination of a point $u_0, \dots u_K \in \mathbb{R}^d$, is called the AFFINE HULL of $u_0, \dots, u_K$
↳ $\langle u_0, \dots u_K \rangle$

Not always points are in general positions!!

EXCLUDE THIS



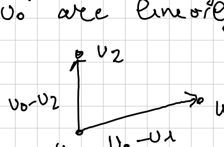
→ ALIGNED

We assume that our points $u_0 \dots u_K$ are AFFINELY INDEPENDENT

$$\left.\begin{array}{ll} \sum \lambda_i u_i = \sum p_i u_i \\ \sum \lambda_i = 1 \quad \sum p_i = 1 \end{array}\right\} \Rightarrow \forall i \ \lambda_i = p_i$$

Def: Affine Independence (intuitive definition, not formal)

The points $u_0, \dots u_K$ are affinely independent iff and only iff $u_1 - u_0, u_2 - u_0, \dots u_K - u_0$ are linearly independent



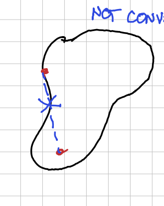
→ Seems to depend on the ORDER of POINTS
↓
TURNS out it doesn't

Def: Convex Set

We are in a vector space $\in \mathbb{R}^d$

A subset $S \subseteq \mathbb{R}^d$ is CONVEX iff $\forall x_1, x_2 \in S$, $\forall \lambda \in [0,1]$ we have that

$(1-\lambda) x_1 + \lambda x_2 \in S$



Def: Compact Set

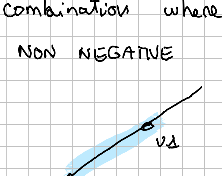
In $\mathbb{R}^d$, a compact set are sets verifying:
- They are closed (The boundary belongs to the set)
- They are bounded

PROPERTIES

When S is compact set, if $\{U_i\}_{i \in I}$ is an infinite family of open subsets of $\mathbb{R}^n$ t.c. $S \subseteq \bigcup_{i \in I} U_i \Rightarrow \{U_{i_1}, \dots, U_{i_K}\}$ finite, t.c. $S \subseteq \bigcup_{r=1}^{K} U_{i_r}$

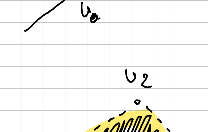
Def: Convex combination

A C.c. of points $\sum_{i=0}^{K} \lambda_i v_i$ is a CONVEX combination where all coefficients λ_i are NON NEGATIVE



AFFINE
$\lambda_0 u_0 + (1-\lambda_0) u_1$
$= u_1 + \lambda_0 (v_0 - v_1)$
↓
$0 \leq \lambda_0 \leq 1$

Now,

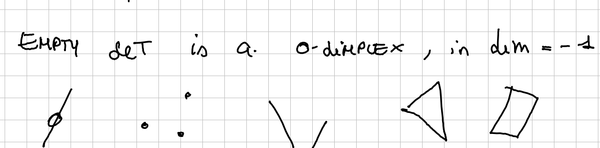


$\lambda_0 u_0 + \lambda_1 u_1 + \lambda_2 u_2$
$\lambda_0, \lambda_1, \lambda_2 \geq 0$
$\sum \lambda_i = 1$

Def k-simplex

Convex hull of k-points that are affinely INDEPENDENT

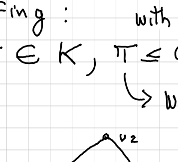
EMPTY SET is a 0-SIMPLEX, in dim = -1



Def: Simplicial Complex

A collection of K-simplexes in $\mathbb{R}^d$, verifying: with $K \neq \emptyset$ $\{\sigma_i\}$ SIMPLEXES

1) $\sigma \in K, \tau \leq \sigma \Rightarrow \tau \in K$
↳ we have many possible

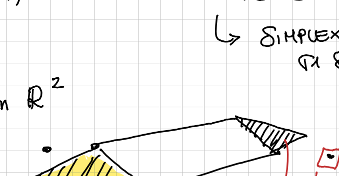


$\sigma = \langle v_0, u_1 u_2 \rangle$ → Convex hull of

$\tau = \langle u_0, u_2 \rangle$ convex hull but also FACE of σ

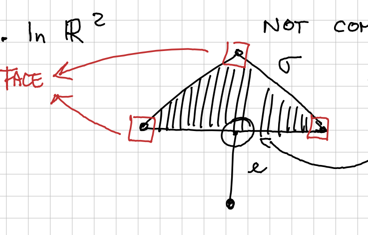
2) $\sigma_1, \sigma_2 \in K \Rightarrow \sigma_1 \cap \sigma_2 \in K$
↳ SIMPLEX & A FACE of σ_1 & σ_2

Ex. In $\mathbb{R}^2$



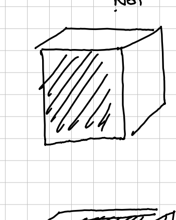
SIMPLICIAL COMPLEX

Ex. In $\mathbb{R}^2$ NOT COMPLEX



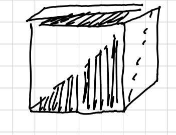
$\sigma \cap e$ is NOT A FACE!

Ex. In $\mathbb{R}^2$ NOT COMPLEX



→ Squares are not legitimate complexes

Ex. In $\mathbb{R}^2$

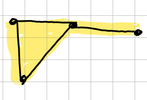


TDA - Lection 2 10/1

Def: Body of a Geometric Simpl. Complex

$|K| = \bigcup_{\sigma \in K} \sigma$ (body of K)

$\cap$ $\mathbb{R}^d$ → COMPACT



The body is ∞ points in $\mathbb{R}^2$

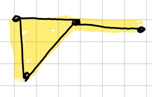
$\emptyset$ 4points 4edges TRIANGLE
1 + 4 + 4 + 1

Def: Subcomplex L of K

Complex L s.t $\sigma \in L \Rightarrow \sigma \in K$
$(L \leq K)$

Def: J-skeleton of K

The collection of all simplexes of K having dim $\leq j$



1-skeleton $j = 1$
↓
$\emptyset$, the points, the edges
(-1) (0) (1)

Def: Abstract Simplicial Complexes

$\{\emptyset\} \neq A = \{\alpha_i\}_{i \in I}$ (Finite) t.c.
↳ NOT ONLY
$\forall \alpha_i \subseteq \mathbb{R}^d$ verifying:

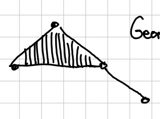
1) $\beta \subseteq \alpha \in A \Rightarrow \beta \in A$ → derives from finitiness of A & α_i

There exist a trivial property (NOT REQUIRED):
- $\alpha_1, \alpha_2 \in A \Rightarrow \alpha_1, \alpha_2$ is finite
- $\alpha_1 \cap \alpha_2 \subseteq \alpha_1$
- $\alpha_1 \cap \alpha_2 \subseteq \alpha_2$

(like geometric complex)

What is an Abstract Simplex?

Geometric Complex



Abstract Complex

$\{\emptyset, \{Alice\}, \{Paul\}, \{Bob\}, \{Alice, Paul\}, \{Alice, Bob\} \{Paul, Bob\}\}$ → no Distance!

Is there a connection between Abstract & Geometric?

Def: Subcomplex abstract L

Subcollection L_A of K_A t.c.
$\sigma \in L_A \Rightarrow \sigma \in K_A$

Def: Isomorphism between A & B (Abstrac)

$\varphi : Vert(A) \rightarrow Vert(B)$
↓
bijection
$\begin{pmatrix} \alpha \in A \Rightarrow \varphi(\alpha) \in B \\ \beta \in B \Rightarrow \varphi^{-1}(\beta) \in A \end{pmatrix}$

Ex. $B' = \{\emptyset, \{Alice'\}, \{Bob'\}, \dots\}$
$B = \{\emptyset, \{Alice\}, \{Bob\}, \dots\}$
$\varphi : B \rightarrow B$ trivial isomorphism

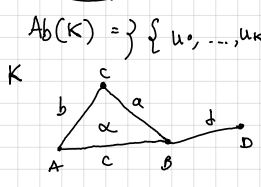
Adding $\{Alice, Bob, Paul\}$ wouldn't give an isomorphism anymore!

Geometric & Abstract simplicial complexes

Let K be a g.s.c, we get Ab(K) called SCHEME of K.

$Ab(K) = \{\{u_0, \dots, u_k\} \text{ s.t } \langle u_0, \dots, u_k \rangle \in K\}$

K

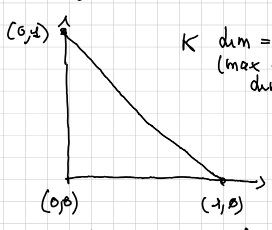


$Ab(K) = \{\emptyset, \{A\}, \{B\}, \{C\}, \{D\}, \{A, B\}, \dots \{A, B, C\}\}$

Can we go from $Ab(K) \rightarrow K$?
↓
Geometric Realization Theorem

What is a REALIZATION of a complex?
↳ The g.s.c. K realizes the abstract simplicial complex B
$\Longleftrightarrow$
$B \cong Ab(K)$
↳ ISOMORPHIC

Let's Try!
$B = \{\emptyset, \{Alice\}, \{Paul\}, \{Bob\}, \{Alice, Paul\}, \{Alice, Bob\}, \{Bob, Paul\}\}$



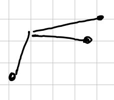
K dim = 1 (max simplex dim)

$Ab(K) = \{\emptyset, \{(0,1)\}, \{1,0\}, \{0,0\}, \{(0,0),(0,1)\}, \dots$

$B \simeq Ab(K)$ → you can assign to each element of B one of Ab(K) (and the opposite)

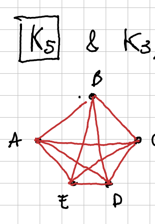
Thm: Geometric Realization Theorem

Let B be an a.s.c. of dim d. B can be realized as a g.s.c. $K \in \mathbb{R}^{2d+1}$



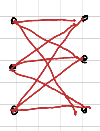
$d = 1$
$\mathbb{R}^{2d+1} = \mathbb{R}^3$

$\boxed{K_5}$ & $K_{3,3}$ don't follow the THEOREM



Adjecent Relation → NON SIMPLEX (There are intersections)

$\boxed{K_{3,3}}$



No isomorphism! INTERSECTIONS

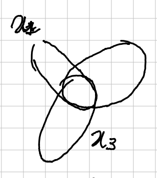
Def: Nerve of a collection of sets

Assume to have a FINITE collection F of open sets:

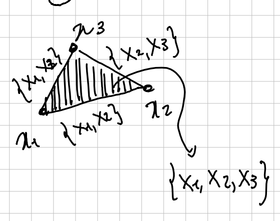
$F = \{x_i\}_{i \in I}$ $(x_i \subseteq Y)$

We call NERVE Nrv(F) the collection of all subsets of F having NON EMPTY INTERSECTION

$\{x_j\}_{j \leq i}$ t.c. $\bigcap x_j \neq \emptyset$



$F = \{x_1, x_2, x_3\}$
$Nrv(F) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}, \{x_1\}, \{x_2\}, \{x_3\}, \emptyset\}$



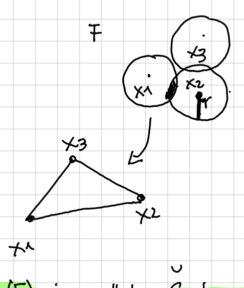
↳ $\bigcap \emptyset \neq \emptyset$ (SI PROVA PER ASSURDO)

EX. Imagine being able to SCAN some object
↓
DISCRETIZE IT

$\{(x_1 x_2, x_3), (y_1, y_2, y_3) \dots\}$ POINT CLOUD in $\mathbb{R}^3$

POINT CLOUDS → SIMPLICIAL COMPLEXES

Let assume $S \subseteq \mathbb{R}^d$, choose $r \geq 0$,
set $F = \{B(x, r) : x \in S\}$



$Nrv(F) = \{\emptyset, \{B(x_1, r)\}, B(x_2, r), B(x_3, r), \{B(x_1, r), B(x_2, r)\}, \{B(x_2, r) B(x_3, r)\}$

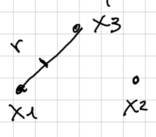
Nrv(F) is called **Čech complex** of S with radius r $\Rightarrow$ Čech(S, r)

dim S = 999.999 with POINT CLOUD OF 100.000.000 points

Vietoris - Rips

$S = \{x_1, \dots x_n\}$ $r \geq \emptyset$

$VR(S, r) = \{$subsets of S having diameter $\leq 2r\}$

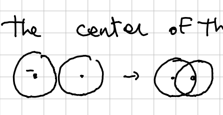


increasing r i get higher dimensions

Theorem: Vietoris - Rips Lemma

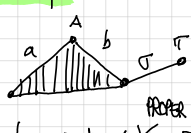
$\check{C}ech(S, r) \subseteq Vietoris\text{-}Rips(S, r) \subseteq \check{C}ech(S, \sqrt{2}r)$

$\frown$ considers the center of the balls



Collapsibility

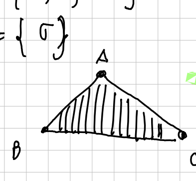
K



$\tau \leq \sigma$ → coface of
τ has 1 coface

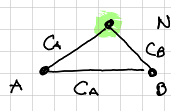
if τ has just 1 PROPER coface σ, we can consider the new complex L obtained from K by removing τ & σ

$CoF(A) = \{a, b, \Delta\}$
$CoF(\tau) = \{\sigma\}$



ELEMENTARY COLLAPSE

A COMPLEX is COLLAPSIBLE iff a sequence of ELEMENTARY COLLAPSES bring to a SIMPLE POINT



NOT COLLAPSIBLE

$\mathbb{Z}_2 = \{0, 1\}$
$(\mathbb{Z}_2, +, \cdot)$

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Def: Quotient Space

Let V be a VECTOR SPACE. Assume that W is a vector subspace of V $W \subseteq V$

$\frac{W}{V}$

Def: Chain Complex (Sequence of VECTOR SPACES)

$\leftarrow V_{p+1} \dots V_p \xrightarrow{\partial_p} \dots V_{p-1} \rightarrow$

∂_p is OMOMORPHISM $\partial_p : V_p \rightarrow V_{p-1}$

Also, it's true that:

$\partial_p \circ \partial_{p+1} \cong \emptyset$

Def: Kernel, Image, Homology of a Chain Complex

Ker ∂_p = set of p-cycles
Im ∂_{p+1} = set of p-boundaries

$\boxed{\frac{Ker\ \partial_p}{Im\ \partial_{p+1}} = H_p(\mathcal{C})}$ HOMOLOGY GROUP OF CHAIN COMPLEX in DEGREE p

$Im\ \partial_{p+1} \subseteq Ker\ \partial_p \Rightarrow \partial_{p+1}(u) \in Ker$
$\partial_p(\partial_{p+1}(u)) = \emptyset$

Def: FORMAL LINEAR COMBINATIONS

$X = \{x_i\}_{i \in I}$, take $a_i \in \mathbb{Z}_i$

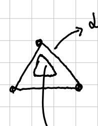
$\sum_{i \in \{0, \dots n\}} a_i x_i$ → Formal way to select some X_i

Lec 3 13/01/'25

CHAIN COMPLEX $\mathcal{C} = (V_p, \nu_p)$ with $\nu_p \circ \nu_{p+1} \approx \emptyset$

$H = \dfrac{\operatorname{Ker} \nu_p}{\operatorname{Im} \nu_{p+1}} \to p$ - HOMOLOGY GROUP

What's the PURPOSE OF THIS → Counting HOLES



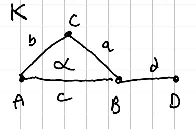
Cavity bounded by dimensional surface

Dim (Hole) = Dim (surface that bounds the hole)

Def: CHAIN COMPLEX C is a sequence of vector spaces V_p over a field K and homomorphism $\nu_p : V_p \to V_{p-1}$

Def: C_p of K

C_p = set of linear combinations of p-simplexes in K



$+ \ \emptyset$

p-simplexes in $d = \emptyset \to C_0 = \{1A + 0B + 0C + 0D,$ B, C, D, A+B, A+C, A+D, B+C, …. A+B+C, …. A+B+C+D $, \emptyset\}$ (in $\mathbb{Z}_2$)

$(1A + 0B + 0C + 0D = A)$

$|C_0| = 2^{n_0} \to$ # of C_0

C_0 is
- COMMUTATIVE
- Can MULTIPLY for 1 & 0 ⇒ VECTOR SPACE in $\dim = n_0$
- ADDITIVE

$C_1 = \{ a, b, c, d, a+b, \dots, \emptyset \}$

$|C_1| = 2^{n_1} \to \dim C_1 = 4$

$C_3 = \{\emptyset\}$ NO SIMPLEX → IT'S THE NULL SPACE

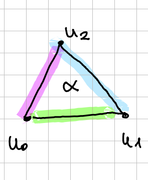
$C_{-1} = \{\emptyset, 1\emptyset\}$ since $C_{-1} = \langle \emptyset \rangle$ $\dim C_{-1} = 2$

(This is the case of reduced HOMOLOGY)

Def: Boundary Map HOMOMORPHISM in VS in linear transformation

Let us consider α, we call boundary in degree 2 of α $\partial_2 \alpha$

$\alpha = \langle u_0, u_1, u_2 \rangle$



$\mathbb{R}^2$

$\partial_2 : C_2 \to C_1$

$\partial_2 \langle u_0, u_1, u_2 \rangle = \langle \hat{u}_0, u_1, u_2 \rangle + \langle u_0, \hat{u}_1, u_2 \rangle + \langle u_0, u_1, \hat{u}_2 \rangle$

REMOVED

In general dimension we have

Def: Boundary Operator

$\partial_p : C_p \to C_{p-1}$

vector space

LINEAR OPERATOR

We take the NATURAL BASIS of C_p (given by all p-simplexes in K) and, for any p-simplex $\sigma \in K$, we set

$\sigma = \langle u_0, \dots, u_p \rangle$

$$\partial_p(\sigma) := \sum_{i=\emptyset}^{p} \langle u_0, \dots, \hat{u}_i, \dots, u_p \rangle$$

Recall: $\partial_{p-1} \circ \partial_p \equiv \emptyset \Rightarrow \partial^2 \equiv \emptyset$

Ex. $\partial_p \langle u_0, u_1, u_2 \rangle = \langle \hat{u}_0, u_1, u_2 \rangle + \langle u_0 \hat{u}_1 u_2 \rangle + \langle u_0 u_1 \hat{u}_2 \rangle$

$\partial_p \partial_p \langle u_0, u_1, u_2 \rangle = \langle \hat{u}_1, u_2 \rangle + \langle u_1, \hat{u}_2 \rangle + \langle \hat{u}_0, u_2 \rangle + \langle u_0, \hat{u}_2 \rangle + \langle \hat{u}_0 u_1 \rangle + \langle u_0 \hat{u}_1 \rangle$

$\langle u_2 \rangle = u_2$

$= u_2 + u_1 + u_2 + u_0 + u_1 + u_0$

$= 2u_0 + 2u_1 + 2u_2 = \emptyset$

In $\mathbb{Z}_2 \Rightarrow 1 + 1 = \emptyset$ $(2 \bmod 1 = 0)$

PROP 13: $\partial^2 \equiv \emptyset$

Proof: $\sigma = \langle u_0, \dots, u_p \rangle$

Consider $\sigma_{ij} = \begin{cases} \langle u_0, \dots \hat{u}_i, \dots \hat{u}_j, \dots u_p \rangle & i < j \\ \text{null } p\text{-chain in } C_{p-2} & i \geq j \end{cases}$

$\partial_{p-1} \circ \partial_p : C_p \to C_{p-2}$

$$\partial_{p-1}(\partial_p \sigma) = \partial_{p-1}\left(\sum_{i=0}^{p} \langle u_0, \dots \hat{u}_i, \dots u_p \rangle\right)$$

$$= \sum_{i=\emptyset}^{p-1}\left(\sum_{j=\emptyset}^{i-1} \langle u_0, \dots, \hat{u}_j, \hat{u}_i, \dots u_p \rangle + \sum_{j=1+i}^{p} \langle u_0, \dots, \hat{u}_i, \hat{u}_j, \dots u_p \rangle\right)$$

$$= \sum_{i=\emptyset}^{p-1}\left(\sum_{j=\emptyset}^{i-1} \sigma_{ji} + \sum_{j=i+1}^{p} \sigma_{ij}\right)$$

use che $i \geq j$

$$= \sum_{i=\emptyset}^{p-1} \sum_{j=\emptyset}^{p} \sigma_{ji} + \sum_{i=\emptyset}^{p-1} \sum_{j=0}^{p} \sigma_{ij} \quad \begin{pmatrix} i-1 \to p \\ i+1 \to \emptyset \end{pmatrix}$$

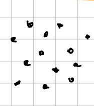
$$= \sum_{j=\emptyset}^{p} \sum_{i=\emptyset}^{p} \sigma_{ij} + \sum_{i=\emptyset}^{p} \sum_{j=\emptyset}^{p} \sigma_{ij}$$

$$= 2 \sum_{i=\emptyset}^{p} \sum_{j=\emptyset}^{p} \sigma_{ij} = \emptyset \Rightarrow \partial^2 \equiv \emptyset$$

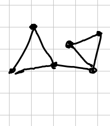
[NOTE: The elements of C_p are called p-chains]

Now $H_p(\mathcal{C}) = \dfrac{\operatorname{Ker} \nu_p}{\operatorname{Im} \nu_{p+1}} = H_p(K)$

Ex. Let's imagine



$\Rightarrow K$



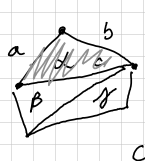
$H_p(K)$ holes in K

Why do homology groups count p-holes?

recall $H_p(K) = \dfrac{\operatorname{Ker} \partial_p}{\operatorname{Im} \partial_{p+1}} = \dfrac{Z_p}{B_p}$ Z_p - p-cycle, B_p → p-boundaries

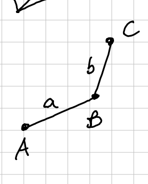
$\operatorname{Ker} \partial_p$ → p-cycles

Ex.



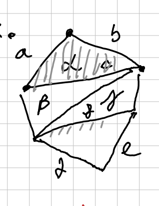
$\partial(a + b + c) = \emptyset$ → cicle

Ex.



$\partial(a + b) = (A + B) + (B + C)$
$= A + 2B + C = A + C$

Ex.



$\partial(d + e + f) = \emptyset$ → still cycles but NOT a BOUNDARY p-HOLE

A Hole is a CYCLE that is NOT a BOUNDARY

1 HOLE OR 2 HOLES ? ⇒ 1

IF the Ker has 2 cycles with a common boundary

HOLES are equivalent

How to compute homology groups?

We will use linear algebra to compute homology groups

$n_p :=$ number of p-simplexes $(= \dim C_p)$ (p-CHAIN)
$z_p := \dim \text{of } Z_p = \dim \operatorname{Ker} \partial_p$ (p-CICLES)
$b_p := \dim \text{of } B_p = \dim \operatorname{Im}(\partial_{p+1})$ (p-BOUNDARIES)
$\beta_p := \dim H_p(K) = \dim \dfrac{Z_p}{B_p}$ (p-HOLES)

Computing $\beta_p \iff$ Computing Homology Group (However it's a VECTOR SPACE)

PROP β_p

$\beta_p = n_p - b_p - b_{p-1}$

PROOF: We have 2 linear maps

$\pi : Z_p \to \dfrac{Z_p}{B_p} = \operatorname{Im} \quad \pi(z) := [z]$ take category

$\dim Z_p = \dim \operatorname{Ker} \pi + \dim \operatorname{Im} \pi$

by def

$z_p = b_p + \beta_p$ (A)

Now consider:

$\partial_p : C_p \to C_{p-1}$

$\dim C_p = \dim \operatorname{Ker} \partial_p + \dim \operatorname{Im} \partial_p$

$n_p = z_p + b_{p-1}$ (B)

Putting together A & B:

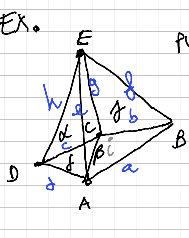
$n_p = b_p + \beta_p + b_{p-1}$

$\beta_p = n_p - b_p - b_{p-1}$

Now $b_p = \dim \operatorname{Im} \partial_{p+1}$

∂_{p+1} is represented by MATRIX A_{p+1} with respect to the N.b. of C_{p+1} & C_p, than i need the RANK of that MATRIX

Ex.



PYRAMID WITH OPEN BASIS, 4 TRIANGLES + 1 segment

$H_p(K)$?

$\beta_p(K) = n_p(K) - b_p(K) - b_{p-1}(K)$

$n_0 = 5$
$n_1 = 9$ $\quad n_p = 19$
$n_2 = 4$
$n_{-1} = 1$

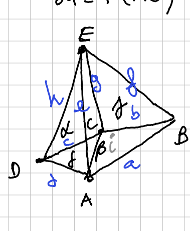
(REDUCED HOMOLOGY) THIS GIVES

$\partial A = \langle \emptyset \rangle$ → NOT EMPTY, BUT THE empty set!!

A_2 represents ∂_2 $\quad \partial_2 : C_2 \to C_1$ $\quad b_1 = r(A_2)$

$$A_2 = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array}$$

$$A_2 = \left[\begin{array}{cccc} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ 1 & & & 1 \\ & & & \end{array}\right] \begin{array}{c} a \\ b \\ c \\ d \\ e \\ F \\ g \\ h \\ i \end{array}$$



MOVE 1 COLUMN & get 4 PIVOTS

$$A_2 = \begin{array}{cccc} \beta & \gamma & \delta & \alpha \end{array}$$

$$\left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & 1 \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{array}\right] \begin{array}{c} a \\ b \\ c \\ d \\ e \\ F \\ g \\ h \\ i \end{array}$$

Now $b_n = r(A_3) = \emptyset$ $\quad B_2 = \operatorname{Im} \partial_3 \subseteq \partial(C_3)$, $C_3 = \emptyset$ → NO 3-SIMPLEX

$b_{-2} = \dim B_{-2} \leq \dim C_{-2} = \emptyset$

So
$b_{-2} = \emptyset$
$b_{-1} = 1$ → $\partial C_0 = \{\emptyset, 0\}$ [REDUCED HOMOLOGY]
$b_0 = ?$
$b_1 = 4$
$b_2 = \emptyset$
$b_3 = \emptyset$

Let's compute b_0!

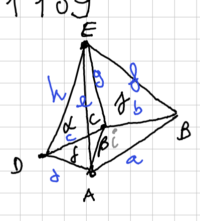
$b_0 = \operatorname{rank}(A_1)$

$(a + b + c)$

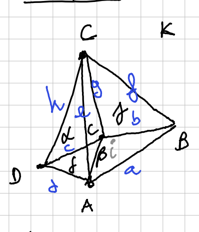
$$A_1 = \begin{array}{c|ccccccccc} & a & b & c & d & e & F & g & h & i \\ \hline A & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ B & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ C & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ D & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ E & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}$$

$r(A_1) = 4$

MUST MOVE COLUMNS



Lec 4 - TDA 15/01/24



$\beta_p = n_p - b_p - b_{p-1}$

$n_p = 19$ $\left(\begin{matrix} n_2 = 4 & n_1 = 9 & n_0 = 5 \\ n_{-1} = 1 \end{matrix}\right)$

$b_p = \dim B_p = \dim \mathrm{Im}\, \partial_{p+1}$

$A_p \approx \partial_p \rightarrow b_p = \mathrm{rank}(A_{p+1}) \nearrow$ $b_{-1} = 1$, $b_0 = 4$, $b_1 = 4$, $b_2 = \emptyset$, $\vdots = \emptyset$

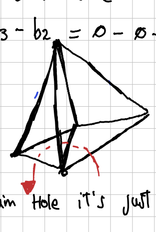
$\beta_2 \simeq n_2 - b_2 - b_1 = 4 - \emptyset - 4 = \emptyset \Rightarrow$ TRIVIAL SPACE

$\beta_1 \simeq n_1 - b_1 - b_0 = 9 - 4 - 4 = \underline{\underline{1}}$ $H_1(K) \in \mathbb{Z}_2$

$\beta_0 \simeq n_0 - b_0 - b_{-1} = 5 - 4 - 1 = \emptyset$

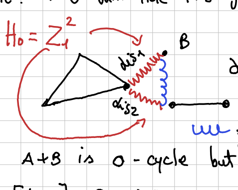
$\beta_{-1} \simeq n_{-1} - b_{-1} - b_{-2} = 1 - 1 - 0 = \emptyset$

$\beta_3 \simeq n_3 - b_3 - b_2 = 0 - \emptyset - \emptyset = \emptyset$



ONE HOLE ONLY (Same class of equivalence)

NOTE: a 0-dim Hole it's just a DISCONNECTION



$\partial(A+B) = \partial A + \partial B$

$\emptyset + \emptyset = 2\emptyset = 0$

A+B is 0-cycle but NOT A 0-BOUNDARY

$[A+B] \in \frac{Z_0}{B_0}$

class $\neq \emptyset$

Reduced Simplicial Homology VS Classical Siml. Homology

RSH	CSH
$b_{-1} = 1$	$b_{-1} = \emptyset$
HERE $\partial_0 : C_0 \rightarrow \tilde{C}_{-1} \cong \mathbb{Z}_2$	$\partial_0 : C_0 \rightarrow C_{-1} = \emptyset$
WITH	WITH:
$\partial_0 P = \emptyset$ (empty set) $\neq$, simplex	$\partial_0 P = 0$ (null chain)
$\boxed{\tilde{b}_{-1} = \dim \mathrm{Im}\, \tilde{\partial}_0 = 1}$	$\boxed{b_{-1} = \dim \mathrm{Im}\, \partial_0 = \emptyset}$

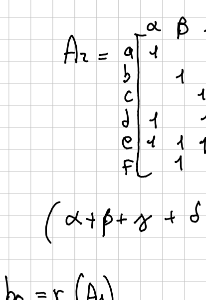
$\mathbb{Z}^2$: space generated by the empty set

$\tilde{\beta}_0 = n_0 - b_0 - \tilde{b}_{-1} < \beta_0 = n_0 - b_0 - b_{-1}$

$\boxed{\beta_0 - \tilde{\beta}_0 = 1}$

Counts # of disconnection — Counts # of connected components (disconnections + 1)

EX.



$n_2 = 4$ b_2
$n_1 = 6$ b_1
$n_0 = 4$ b_0

- $b_2 = r(A_3)$ but A_3 is empty
  $b_2 \subseteq \dim B_2 \approx \dim \mathrm{Im}\, \partial_3$ but $C_3 = \emptyset \Rightarrow \mathrm{Im}\, \partial_3 = \emptyset$

$b_1 = r(A_2)$

$$A_2 = \begin{array}{c|cccc} & \alpha & \beta & \gamma & \delta \\ \hline a & 1 & & & 1 \\ b & & 1 & & 1 \\ c & & & 1 & 1 \\ d & 1 & & 1 & \\ e & 1 & 1 & 1 & \\ f & & 1 & & \end{array} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & & & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & \\ & 0 & 0 & 1 \end{bmatrix} = r(A_2) = 3$$

$(\alpha + \beta + \gamma + \delta = \emptyset$ is a cycle in $b_1)$

$b_0 = r(A_1)$

$$\begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline A & 1 & 0 & 1 & 1 & 1 & 0 \\ B & 1 & 1 & 0 & 0 & 0 & 0 \\ C & 0 & 1 & 1 & 0 & 1 & 1 \\ D & 0 & 0 & 0 & 1 & 0 & 1 \end{array} \Rightarrow R(A_1) = 3$$

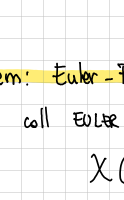
actually reduce it!

Now:

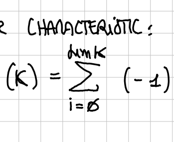
- $\beta_2 = n_2 - b_2 - b_1 = 4 - 0 - 3 = 1$ $= H_2(K) \in \mathbb{Z}_2$
- $\beta_1 = n_1 - b_1 - b_0 = 6 - 3 - 3 = \emptyset$
- $\beta_0 = n_0 - b_0 - b_{-1} = 4 - 3 - 1 = \emptyset$

We have 1 hole that's THE CAVITY inside the prysm.

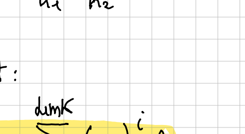
Theorem: IF L simplicial complex is obtained from K by applying an ELEMENTARY COLLAPSE, then $H_p(L) = H_p(K)$ $\forall p \in \mathbb{Z}$



STEP 1: Remove σ & τ
STEP 2: Repeate ∀ edge



STEP 3: REMOVE A & B and coface



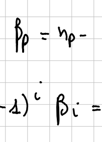
$H_p(L) = H_p(K)$

Theorem: Euler-Poincaré Theorem

We call EULER CHARACTERISTIC:

$$\chi(K) = \sum_{i=\emptyset}^{\dim K} (-1)^i n_i$$

Ex.



$\chi(K) = \underset{n_0}{4 \text{ VERTEX}} - \underset{n_1}{6} + \underset{n_2}{4} = 2$

The Theorem states that:

$$\chi(K) = \sum_{i=\emptyset}^{\dim K} (-1)^i n_i = \sum_{i=\emptyset}^{\dim K} (-1)^i \beta_i$$

WE USE CLASSICAL SIMPLICIAL HOMOLOGY

$\chi(K) = \beta_0 - \beta_1 + \beta_2 = 1 - 0 + 1 = 2$

Proof. We know that

$\beta_p = n_p - b_p - b_{p-1}$

$$\sum_{i=0}^{\dim K} (-1)^i \beta_i = \sum_{i=\emptyset}^{\dim K} (-1)^i n_i - \overbrace{\sum_{i=\emptyset}^{\dim K} (-1)^i b_i}^{A} - \underbrace{\sum_{i=\emptyset}^{\dim K} (-1)^i b_{i-1}}_{B}$$

Now $j \rightarrow i-1$

$$B \Rightarrow \sum_{j=-1}^{\dim K - 1} (-1)^{j+1} b_j = -\sum_{i=-1}^{\dim K - 1} (-1)^i b_j$$

$$\dots = \sum_{i=\emptyset}^{\dim K} (-1)^i n_i - \overbrace{\cancel{\sum_{i=\emptyset}^{\dim K - 1} (-1)^i b_i} - \cancel{(-1)^{\dim K} b_{\dim K}}}^{A} + \cancel{\sum_{i=\emptyset}^{\dim K - 1} (-1)^i b_i} + \cancel{(-1)^{-1}} b_{-1}$$

$b_{-1} = 0$ in CLASSICAL HOMOLOGY

$$= \sum_{i=\emptyset}^{\dim K} (-1)^i n_i$$

Def. GROUP $(G, *)$

Set of elements that follow some properties:

1) $e * g = g * e$ $g, e \in g$ e identity
2) $(g_1 * g_2) * g_3 = (g_2 * g_3) * g_1 \in g$
3) $\forall g_i \in G$, $\exists g_{-i}$ s.t. $g * g_{-1} = g_{-1} * g = e$

ABELIAN GROUP (COMMUTATIVE) → order of operation doesn't matter

ex. $(G, \circ) \rightarrow$ isometries on the plane

Def. Pseudometric

It's a metric without the property:

- $d(x_1, x_2) = \emptyset \Rightarrow x_1 = x_2$

Def. Group Action $(G, \cdot)$

Take a set X, $x \in X$, $g \in G$

$\boxed{g * x := g(x) \text{ (in } \mathbb{R})}$

The ACTION is a MAP (this is a LEFT-GROUP ACTION)

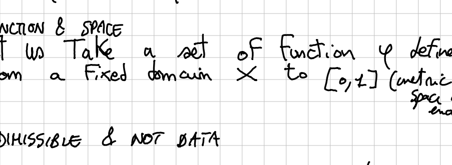
$\circledast : G \times X \rightarrow X$

(HOMORPHISM)

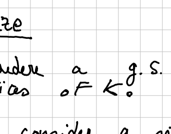
with property:

1) $e * g = g$
2) $g_2 * (g_1 * x) = (g_1 \circ g_2) * x$

GENEO



POINT CLOUD — FUNCTION — GRAPH



$\{g_i * x\}$ ORBIT OF A GROUP

→ FUNCTION & SPACE
Let us Take a set of Function φ defined from a Fixed domain X to [0,1] (metric space is enough)

→ ADMISSIBLE & NOT DATA

→ CHOOSE a GROUP G ACTING o φ by RIGHT COMPOSITION

$\varphi \overset{g}{\longmapsto} \varphi \circ g$

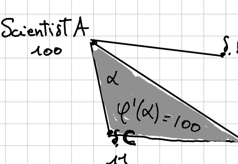
FORMALIZE

Consider a g.s.c. K and its set of vertices of K^0

Now consider a set of Functions Φ'

$\Phi' : V \rightarrow \mathbb{R}$

if $\varphi' \in \Phi'$ $(\varphi' : V \rightarrow \mathbb{R})$

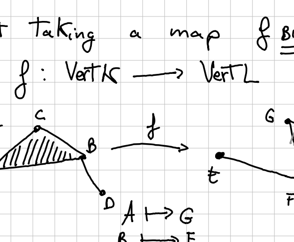


$\sigma = \langle u_0, \dots, u_K \rangle$ $\varphi(\sigma) = \max(\varphi'(u_0), \dots \varphi'(u_K))$

So $\varphi(\alpha) = 10$
$\varphi(b) = 10$ we extended the value of the max $\varphi'(u_i)$
$\vdots$

We obtain Φ by extending Φ'!

EX.



(We need a group acting on this functions)

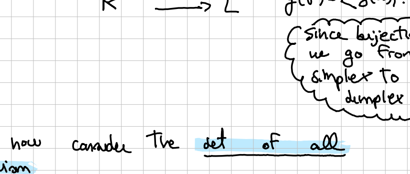
Actually we need ISOMORPHISM acting on wr g.s.c.

$K \longmapsto L$

We start taking a map f BIJECTIVE:

$f : \mathrm{Vert}\, K \longrightarrow \mathrm{Vert}\, L$

Ex.



$A \mapsto G$
$B \mapsto F$
$C \mapsto H$
$D \mapsto E$

I extend f:

$$\begin{array}{ccc} \mathrm{Vert}\, K & \longrightarrow & \mathrm{Vert}\, L \\ & \downarrow g & \\ K & \longrightarrow & L \end{array}$$

$\sigma = \langle u_0, \dots, u_K \rangle$

$g(\sigma) = \langle f(u_0), \dots f(u_K) \rangle$

(since bijection we go from simplex to simplex)

Let us now consider the set of all isomorphism

$g : K \longrightarrow K$

s.t. $\varphi \in \Phi \longrightarrow \varphi \circ g \in \Phi \rightarrow \varphi \circ g_{-1} \in \Phi$ (preserves Φ)

We call this set $\mathrm{Iso}_\Phi(K)$

Lez 5 - TDA 17/01

PSEUDODISTANCE - WHAT DO WE NEED? ①, ②, ③ & ④

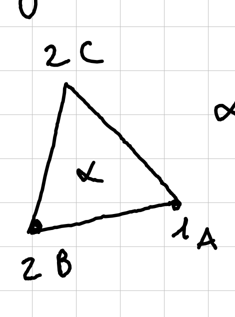
① Let's get a g.s.c K, and a function φ

$$\varphi' : \mathrm{Vert}K \longrightarrow \mathbb{R}$$

We extend this to:

$$\varphi : K \longrightarrow \mathbb{R} \qquad (\varphi \in \Phi \text{ MEASURES})$$

Using that $\varphi(K) = \max(\varphi'(u_0), \dots \varphi'(u_K))$



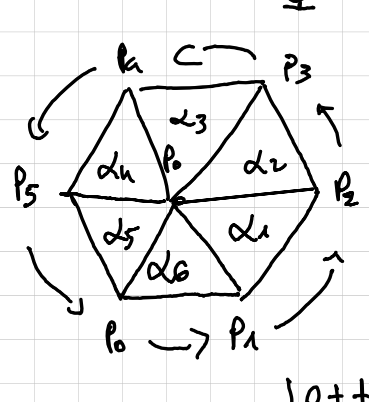
$\alpha = \max(\varphi'(A), \varphi'(B), \varphi'(C))$
$= 2$

② Now consider the set of the isomorphism on K

$$f : K \longrightarrow K$$

$$\langle u_0, \dots U_K \rangle \in K \iff \langle f(u_0), \dots f(u_K) \rangle \in K$$

③ A group s.t $\mathrm{Iso}_\Phi(K) \subseteq \mathrm{Iso}(K)$

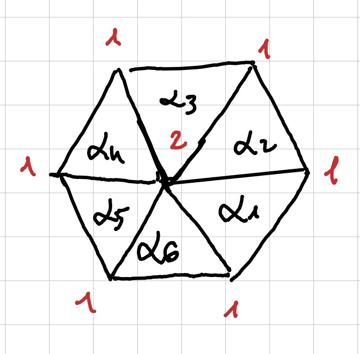


K

The set of ISOMORPHISM : $\{$Rotation of $p_1 \dots p_6$, $(p_0 \to p_0)$ Reflection on symmetries$\}$
$\Downarrow$
$\mathrm{Iso}(K)$

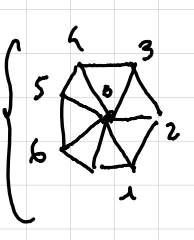
If $\mathrm{Iso}_\Phi(K)$:

$$\begin{cases} \varphi'(p_1) = 1 \\ \varphi'(p_2) = 1 \\ \varphi'(p_3) = 1 \\ \vdots \\ \varphi'(p_0) = 2 \end{cases}$$



$\Rightarrow \mathrm{Iso}_\Phi(K) = \mathrm{Iso}(K)$

But if $\mathrm{Iso}_\Phi(K)$:



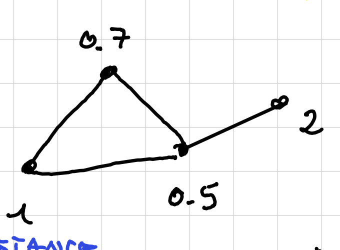
$\Rightarrow \mathrm{Iso}_\Phi \neq \mathrm{Iso}(K)$

④ Assume that a pair (Φ, G) with $G \subseteq \mathrm{Iso}_\Phi(K)$ is given.

We call this pair a SIMPLICIAL PERCEPTION PAIR

$$(\varphi \circ g \;\&\; \varphi \circ g^{-1} \in \Phi)$$

PSEUDOMETRIC D_V on $V = $ Vert of K



Not using the METRIC in $\mathbb{R}^2$!!
Derives from instrument

1) DISTANCE $D_V(v_1, v_2) = \sup_{\varphi \in \Phi} |\varphi(v_1) - \varphi(v_2)|$ METRICS FOR VERTICES

2) P. DIST $D_G(g_1, g_2) = \sup_{\varphi \in \Phi} \|\varphi \circ g_1 - \varphi \circ g_2\|_\infty$ METRICS FOR $g \in \mathrm{Iso}_\Phi(K)$

3) P. DIST $D_\Phi(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|_\infty = \max_{v \in V} |\varphi_1(v) - \varphi_2(v)|$ METRICS FOR SIGNAL SPACE

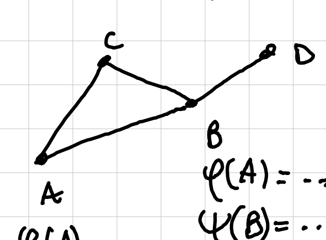
TRIVIAL (from isometry)

The action of any $G \subseteq \mathrm{Iso}_\Phi(K)$ preserves D_Φ, D_V, and D_G.

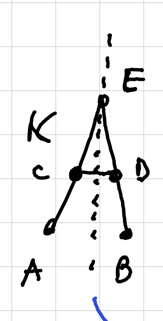
Def: NATURAL PSEUDODISTANCE

φ, ψ functions → Don't compare φ, ψ only, but them under the action of the group G

$$\varphi, \psi : K \longrightarrow \mathbb{R}$$



$\varphi(A) = \dots$
$\psi(B) = \dots$
$\varphi(A) = \dots$
$\psi(\psi) = \dots$

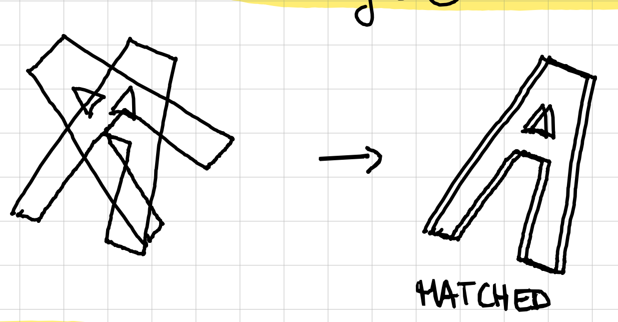


$\Phi = \mathbb{R}^{\mathrm{Vert}K}$

$f : K \to K$
$f(A) = B$
$\vdots$
$f(E) = E$
$\Big\}\ \mathrm{Iso}_\Phi(K)$

The N.P.D is defined as:

$$d_G(\varphi, \psi) = \min_{g \in G} \|\varphi - \psi \circ g\|_\infty$$



MATCHED

PROOF IT'S SYMMETRIC:

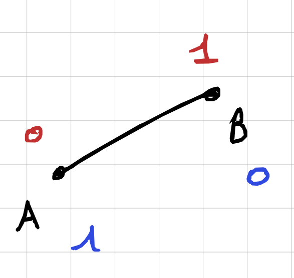
$$d_G(\psi, \varphi) = \min_{g \in G} \|\psi - \varphi \circ g\|_\infty = \min_{g^{-1} \in G} \|\psi - \varphi \circ g^{-1}\|$$

(use $\|\alpha - \beta\|_\infty = \|\alpha \cdot h - h \cdot \beta\|_\infty$)

$$= \min_{g^{-1} \in G} \|\psi \circ g - \varphi \circ g^{-1} \circ g\|$$

$$= \min_{g^{-1} \in G} \|\psi \circ g - \varphi\| = \min_{g \in G} \|\varphi - \psi \circ g\| \quad \blacksquare$$

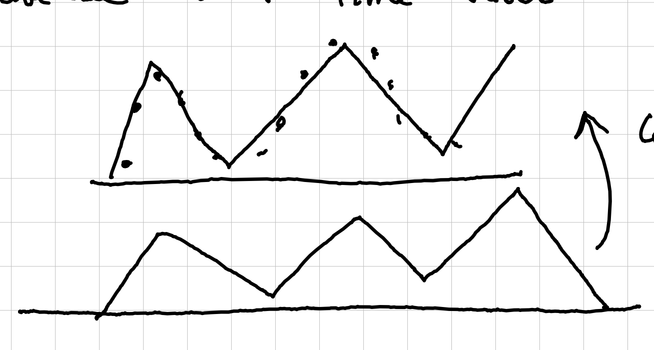
Ex.



$G = \{A \to A, A \xrightarrow{s} B, B \xrightarrow{s} A, B \to B\} = \{\mathbb{1}, \mathrm{SYM}\}$

$\|\varphi_1 - \varphi_2 \circ s\|_\infty = \emptyset \to \varphi_1 \neq \varphi_2$ ma $\emptyset$

Very useful with time series



Can be compared IF TRASLATED $d_T(\varphi, \psi)$

What happens when we increase the group?

$$\mathrm{Id}_V \longrightarrow \mathrm{Iso}_\Phi(K)$$

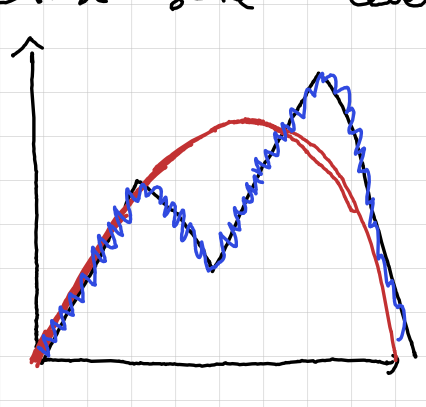
1) G increases, d_G DOES NOT INCREASE

2) d_G is STRONGLY INVARIANT

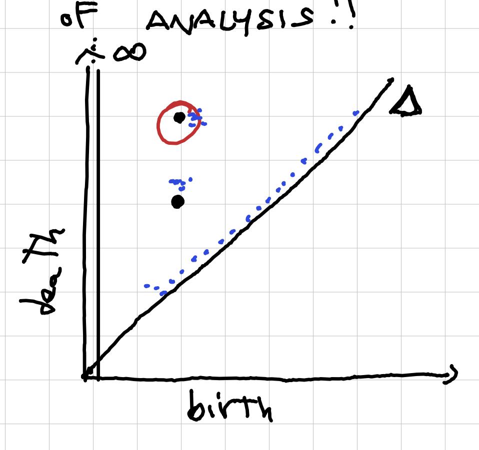
$$d_G(\varphi \circ g_1, \psi \circ g_2) = d_G(\varphi, \psi)$$

EX.

Choose some level of ANALYSIS!!



Noise



1) Δ is 1 ABSTRACT POINT WITH MULTIPLE MOLTIPLICITY

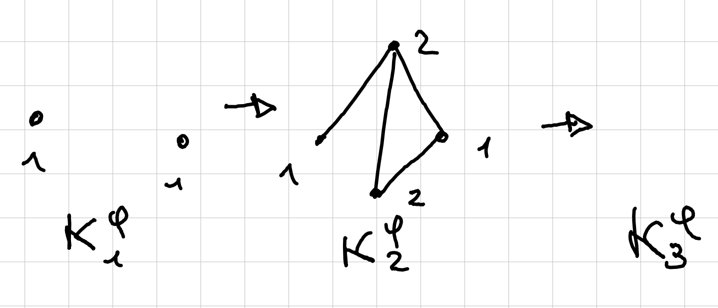
2) Line to ∞ means there is a connected component

3) Point O gives behavior

Closer to diagonal ⇒ MORE DETAILS

Let's consider a function $\varphi : K \to \mathbb{R}$ (extension of φ')

We define $K_\mu^\varphi = \{\sigma \in K : \varphi(\sigma) \leq \mu\}$



Sequence of K → Sequence of $H(K)$

Def: Persistence Module

Collection of vector spaces $\{V_t\}_{t \in \mathbb{R}}$ with

$$i_{s,t}^\varphi : V_s \longrightarrow V_t \quad \text{with } s \leq t$$

s.t. 1) $i_{r,t}^\varphi = i_{s,t}^\varphi \circ i_{r,s}^\varphi$ [FUNCTIONALITY]

2) $i_{t,t}^\varphi = id_{V_t}$ [IDENTITY]

Now, consider the vector spaces:

$$\underset{V_t}{\underset{\shortparallel}{H_p}}(K_t^\varphi) \quad \text{for } t \in \mathbb{R}$$

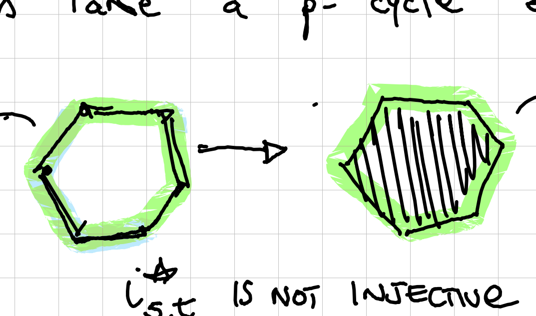
Let's consider, for $s \leq t$, the INCLUSION:

$$i_{s,t} : K_s^\varphi \longrightarrow K_t^\varphi \qquad (K_1 \subset K_2)$$

This map induces a linear map:

$$i_{s,t}^* : H_p(K_s^\varphi) \longrightarrow H_p(K_t^\varphi)$$

Ex. Let's take a p-cycle $z \in C_p(K_s^\varphi)$



cycle $H_1(K_s^\varphi)$

→ boundary $b \in H_1(K_t^\varphi)$

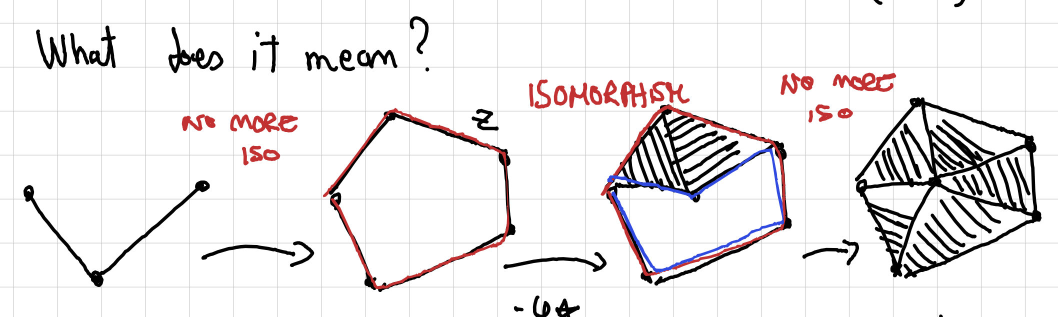
$i_{s,t}^*$ IS NOT INJECTIVE

[In our construction $i_{r,t}^{\varphi *} = i_{s,t}^{\varphi *} \circ \varphi_{r,s}^{t *}$]

Def: Persistence Homology Group

$H_p^\varphi(u, v) := \mathrm{Im}\, i_{u,v}^{\varphi *}$ $(u < v)$ is the PERSISTENC HOMOLOGY GROUP OF φ, COMPUTED AT (u, v)

What does it mean?



- $t_1 \Rightarrow$ Homology group changes, a cycle is born at t_1 $([z] \in H_1(K_{t_1}^\varphi))$

$$H_1(K_{t_0}^\varphi) \xrightarrow{i_{t_0,t_1}^{*\varphi}} H_1(K_{t_1}^\varphi) \text{ A BIRTH}$$

$\mathbb{Z}_2$

- $t_3 \Rightarrow H_1(K_{t_3}^\varphi) = \emptyset$ (BOUNDARY) A DEATH

Def. Persistent Betti number Function (RANK OR INVARIANT)

$$\beta_p^\varphi(u, v) = \dim H_p^\varphi(u, v) \text{ of } \varphi \text{ in degree } p.$$

Lec 6 - TDA

Persistence module

$H_p(K_u^\varphi) \xrightarrow{i^{\varphi *}_{u,v}} H_p(K_v^\varphi)$ $\quad v \geq u$

with ① Identity
② Functionality

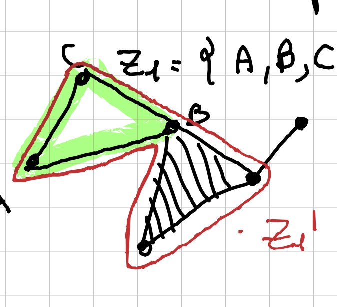
PERSISTENT HOMOLOGY GROUP

$H_p^\varphi(u,v) := \mathrm{Im}\, i^{\varphi *}_{u,v}$

$$H_p(K_u^\varphi) \xrightarrow{i^*_{u,v}} H_p(K_v^\varphi) \supseteq \mathrm{Im}_{u,v}$$

Let's take a set $\{z_1, \dots z_h\}$ of p-cycles in $C_p(K_u^\varphi)$ s.t. $\{[z_1]_u, \dots [z_h]_h\}$ is a basis in $H_p(K_u^\varphi)$

K_4 $\quad z_1 = \{A, B, C\}$ in H_1



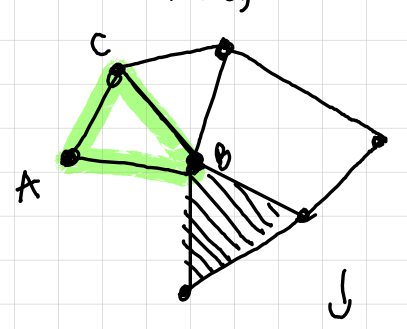
$[z]_u \simeq [z_1']_u$ are equivalent (represent the same class) in $H_1(K_4^\varphi)$ ↳ (time step)

So z_1 is a basis vector in H_1

- Let's now consider the equivalence classes $[z_1]_v$

$[z_1]_v \dots, [z_h]_v$
$\cap$
$H_p(K_v^\varphi)$

Since $i^*_{u,v}$ is **NOT INJECTIVE**, hence $\{[z_1]_v, \dots [z_h]_v\}$ is **NOT a BASIS** but spans $\mathrm{Im}\, i^*_{u,v}$

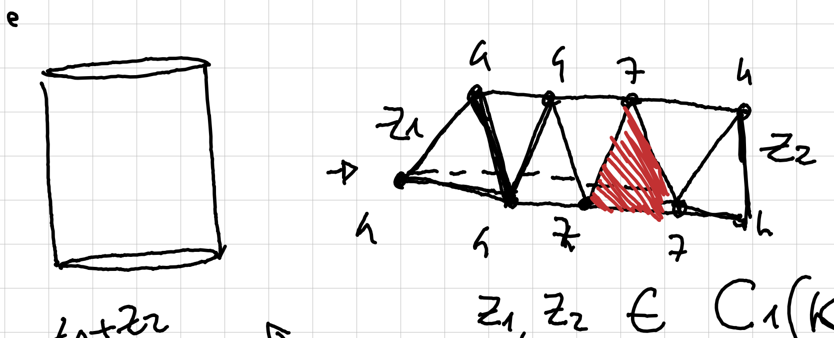


$H_1^\varphi(4,7) = \mathbb{Z}_2$

1-HOLE ALREADY PRESENT in K_4

- $\{[z_1]_u, \dots [z_h]_u\}$ is a basis for $H_p(K_u^\varphi)$
- $\{[z_{i_1}]_v, \dots [z_{i_h}]_v\}$ is a basis for $H_p(K_v^\varphi)$
  ↳ HOLES ALREADY present UNDER v

Ex.



$z_1 + z_2$ can't be obtained as a boundary chain

$z_1, z_2 \in C_1(K_4^\varphi)$

$[z_1]_4 \neq [z_2]_4$ here independent

but $[z_1]_7 = [z_2]_7$ here dependent

Def: Betti dimension properties

We defined $\beta_p^\varphi(u,v) = \dim H_p(u,v^\varphi)$

1) β_p^φ is RIGHT-CONTINUOUS in BOTH its VARIABLES

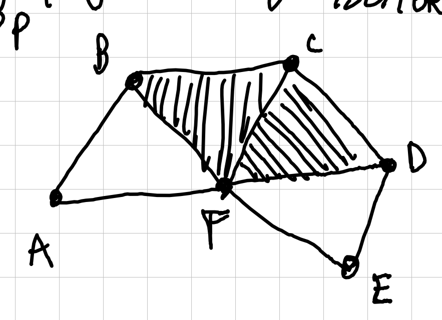
$\cdot (u,v)$
$\cdot (u+\varepsilon, v+\varepsilon)$

$\forall\ \varepsilon \geq \emptyset$ with ε small enough

$$\beta_p^\varphi(u,v) = \beta_p^\varphi(u+\varepsilon, v+\varepsilon)$$

2) $\beta_p^\varphi(u,v)$ is NON-INCREASING in v and NON-DECREASING in u

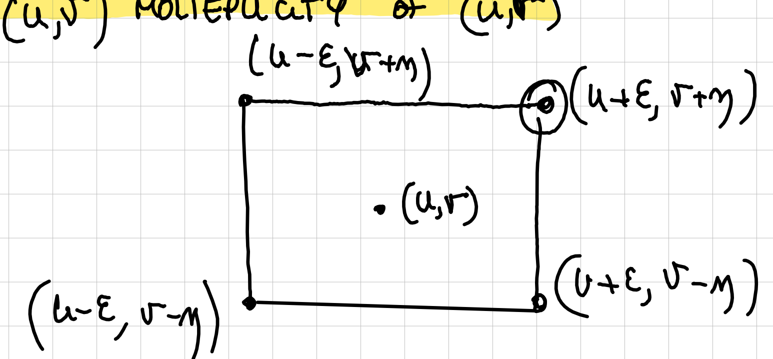
3) $\beta_p^\varphi \equiv \beta_p^{\varphi \circ g^{-1}}$ $\forall$ ISOMORPHISM $g \in \mathrm{Iso}_\Phi(K)$



$g: K \to K$
$A \mapsto E$
$B \mapsto D$
$F \mapsto F$
$C \mapsto C$

$\Rightarrow$ we are just renaming the vertices of K, the β_p^φ does not change

Def: $\mu^\varphi_{\varepsilon,\eta}(u,v)$ MOLTEPLICITY of (u,v)

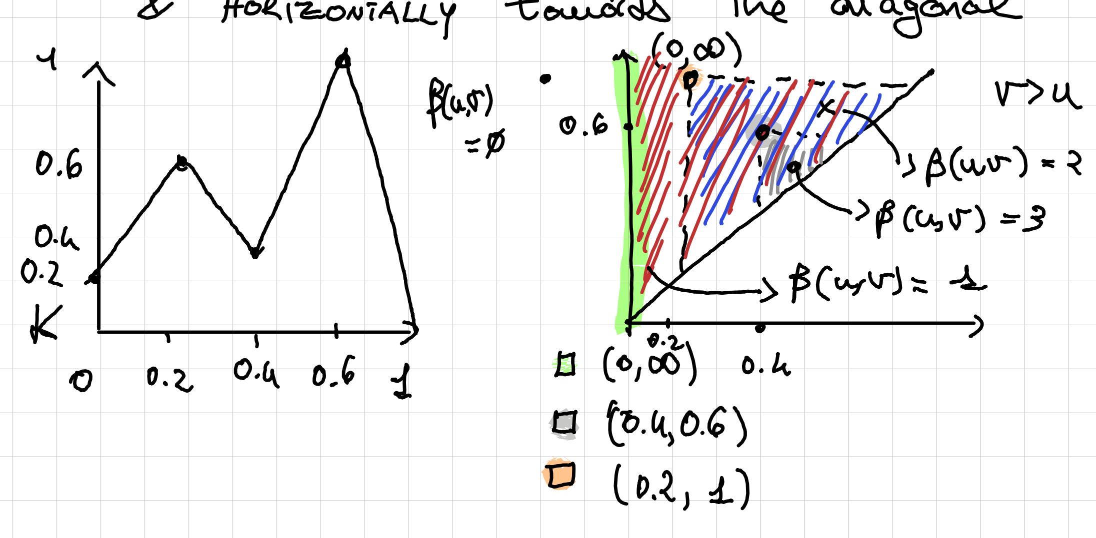


$$\mu^\varphi_{\varepsilon,\eta}(u,v) = \beta(u+\varepsilon, v-\eta) + \beta(u-\varepsilon, v+\eta) - \beta(u-\varepsilon, v-\eta) - \beta(u+\varepsilon, v+\eta)$$

Also, ENLARGING The rectangle $\mu^\varphi_{\varepsilon', \eta'} \geq \mu^\varphi_{\varepsilon,\eta}(u,v)$

Discontinuities of $\beta_p^\varphi(u,v)$

1) The discontinuities of β_p^φ propagate VERTICALLY & HORIZONTALLY towards the diagonal

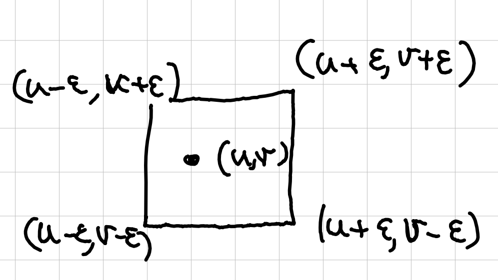


β is the # of tringles superimposed in (u,v)

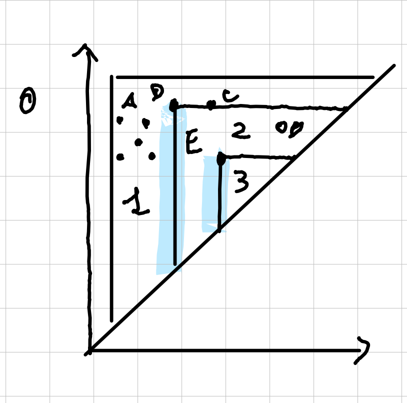
Def. Persistance diagram

β_p defined as above,

if $\lim_{\varepsilon \to \emptyset} \mu^\varphi_{\varepsilon,\varepsilon}(u,v) > 0$
$\downarrow$
(u,v) is a POINT in The PERSISTENT DIAGRAM $Dgm_p(\varphi)$



Ex.



$\mu^\varphi_{u,v}(A) = 1+1-1-1 = \emptyset$

$\mu^\varphi_{u,v}(B) = 2+2-2-2 = \emptyset$

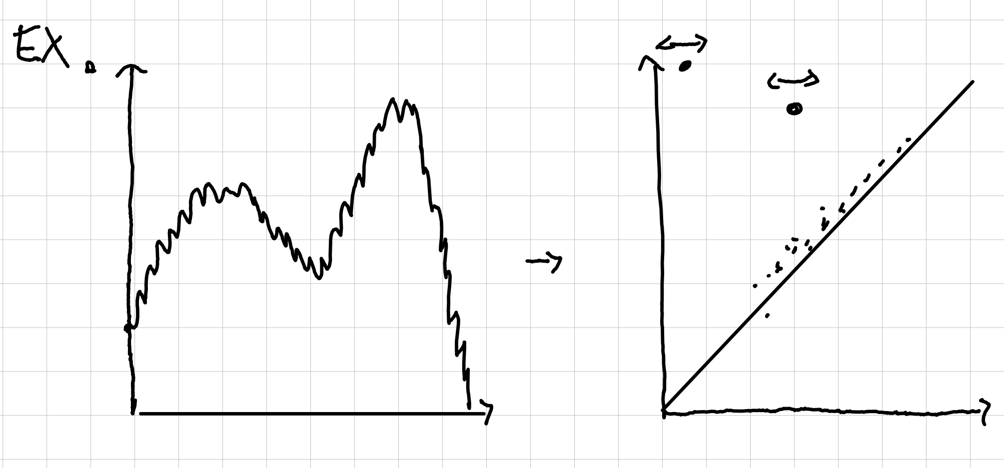
$\mu^\varphi_{u,v}(C) = 2+1-2-1 = 0$

$\mu_{u,v}(D) = 2+1-1-1 = 1$

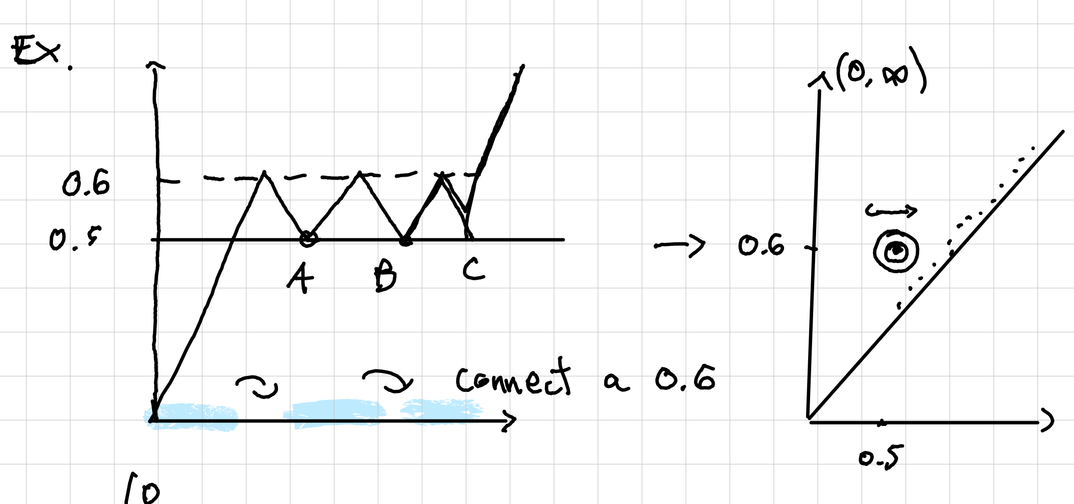
$\mu_{u,v}(E) = 3+2-2-2 = 1$

$$\boxed{\text{Persistent diagram} \iff \beta}$$

Ex.

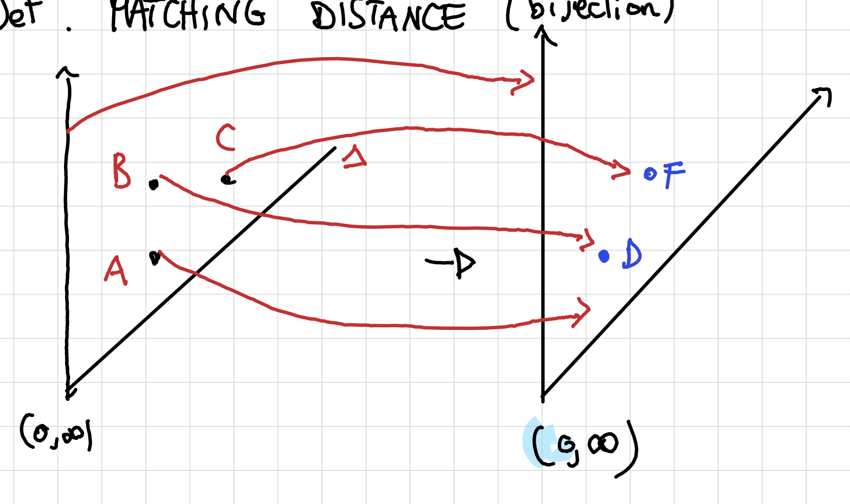


Ex.



(0
A (0.5, 0.6)
B (0.5 0.6)
C (0.5, 0.6)

Def. MATCHING DISTANCE (bijection)

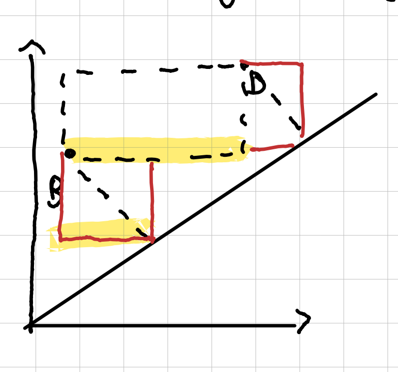


$Dgm(\varphi) = \{(u_1, \infty), (u_2, \infty), A, B, C, \Delta, \dots\}$

$Dgm(\psi) = \{(\bar{u}_1, \infty), (\bar{v}_2, \infty), D, F, \Delta \dots\}$

$\forall$ Bijection, I DEFINE a **Cost**: $\rightarrow$ cost (f)?

$$f: Dgm(\varphi) \to Dgm(\psi)$$



$d(B,D)$?
① Max norm $d_{MAX}(B,D)$
② Move to diagonal first

$\inf d(B,D)$ = MATCHING DISTANCE $\rightarrow$ BOTTLENECK distance

When $\varphi, \psi \in \Phi$

$$d_H(Dgm(\varphi), Dgm(\psi)) \leq d_G(\varphi, \psi)$$ (Stability Theorem)

this itself is difficult to compute

q-TH WASSERSTEIN DISTANCE

$$W_q(X,Y) = \left[\inf_{\eta: X \to Y} \sum_{x \in X} \| x - \eta(x) \|_\infty^q \right]$$

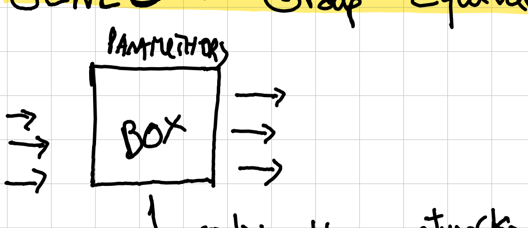
X e Y PERSISTANCE DIAGRAMS

Bottleneck distance

$$W_\infty(X,Y) = \inf_{\eta: X \to Y} \sup_{x \in X} \| x - \eta(x) \|_\infty$$

Lez 7 - TDA 22/01/'24

GENEO - Group Equivariant Non Expansive Operators

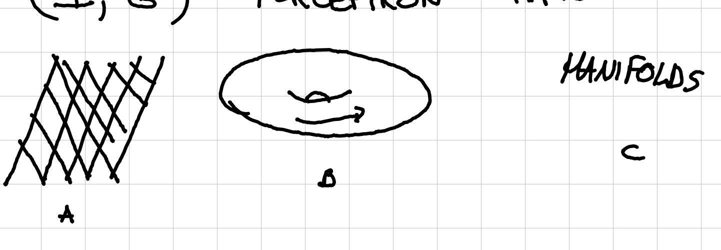


↳ explainable networks

Main points:

→ Data is not so important, but rather (DATA, OBSERVER) ← "a measure"

Ex. (Φ, G) PERCEPTRON PAIR



PROP 41. Any $g \in Iso_\Phi(K)$ is an isometry

Proof. $D_V(x_1, x_2) := \sup_{\varphi \in \Phi} |\varphi(x_1) - \varphi(x_2)|$

Now we want $D_V(g(x_1), g(x_2)) =$

$= \sup_{\varphi \in \Phi} |\varphi(g(x_1)) - \varphi(g(x_2))|$ use $\varphi \circ g \in \Phi$

$= \sup_{\varphi \in \Phi} |\varphi(x_1) - \varphi(x_2)|$ ∎

Recall:

$D_G(x_1, x_2) = \sup_{\varphi \in \Phi} \|\varphi \circ g_1 - \varphi \circ g_2\|_\infty$

if one measurement REVEALS a difference between g_1 & g_2, ⇒ GOOD

Prop. If D_V is a metric, D_G is a metric too.

Def. GEO

Let us consider two simplicial pairs: (Φ, G), (Ψ, H)

Now consider:

1) $F: \Phi \to \Psi$
2) a homomorphism $T: G \to H$

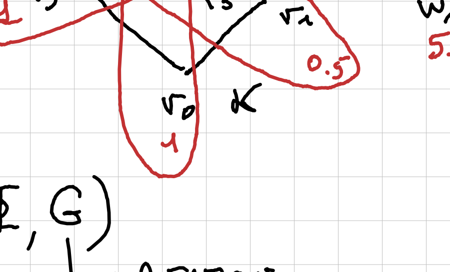
if $F(\varphi \circ g) = F(\varphi) \circ F(g)$ $\forall \varphi, \forall g$

(F, T) is a GEO (Group Equivariant Operators)

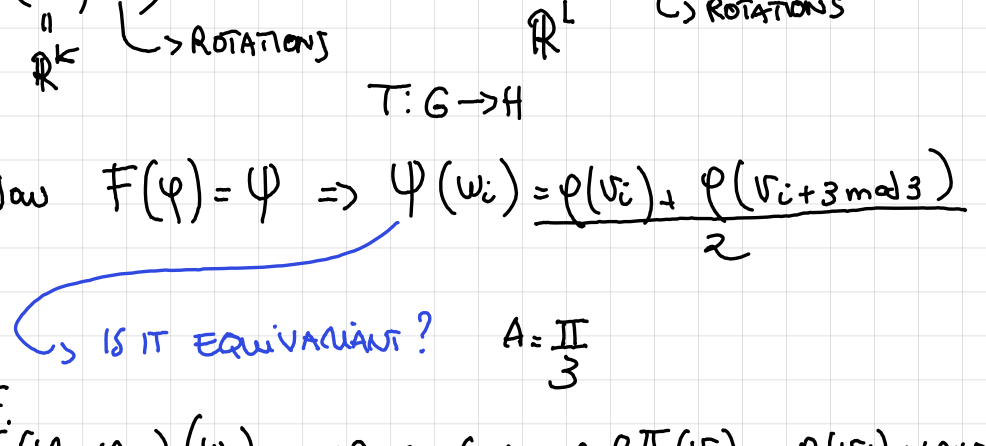
Def. GENEO

if a GEO (F,T) is s.t. F is NON-EXPANSIVE, we say that (F,T) is a group equivariant non expansive operator.

$$D_\Psi(F(\varphi_1), F(\varphi_2)) \le D_\Phi(\varphi_1, \varphi_2)$$



Ex.



$T(\rho_\alpha) = \rho_{2\alpha}$

(Φ, G), $\Phi = \mathbb{R}^K$, G ↳ ROTATIONS

(Ψ, H), $\Psi = \mathbb{R}^L$, H ↳ ROTATIONS

$T: G \to H$

Now $F(\varphi) = \psi \Rightarrow \psi(w_i) = \dfrac{\varphi(v_i) + \varphi(v_{i+3 \bmod 3})}{2}$

↳ IS IT EQUIVARIANT? $A = \frac{\pi}{3}$

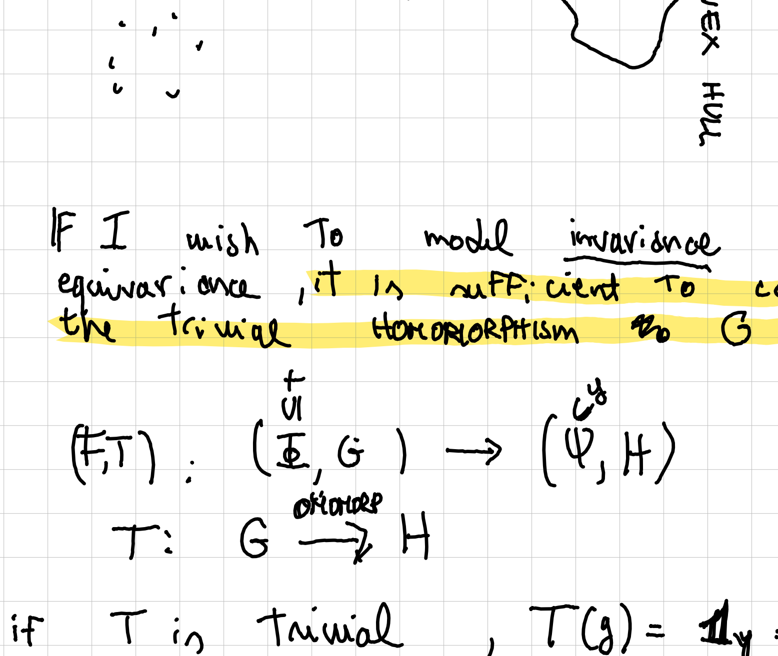
Proof.

$F(\varphi \circ \rho_A)(w_0) = \varphi \circ \rho_{\frac{\pi}{3}}(v_0) + \varphi\, \rho_{\frac{\pi}{3}}(v_3) = \dfrac{\varphi(v_1) + \varphi(v_4)}{2}$

$(F(\varphi) \circ \rho_{\frac{2\pi}{3}})(w_0) = F(\varphi)(w_1) = \dfrac{\varphi(v_1) + \varphi(v_4)}{2}$

$\rho_{\frac{2\pi}{3}}(w_0) = w_1$ This holds $\forall$ point ∎

Geneo's in Topological settings



If I wish to model invariance instead of equivariance, it is sufficient to consider the trivial HOMOMORPHISM of $G \to H$:

$(F,T): (\Phi, G) \to (\Psi, H)$, $\varphi \in \Phi$, $\psi \in \Psi$

$T: G \xrightarrow{\text{HOMOMORP}} H$

if T is trivial, $T(g) = \mathbb{1}_\Psi \Rightarrow$

$F(\varphi \circ g) = F(\varphi) \circ T(g)$

$= F(\varphi) \circ \mathbb{1}_\Psi = F(\varphi)$ ∎

Prop:

Assume that (F,T) is a GENEO

$D_\Phi(F(\varphi_1), F(\varphi_2)) \le D_\Phi(\varphi_1, \varphi_2)$ known

and $(F,T): (\Phi, G) \to (\Psi, H)$

Then:

$$d_H(F(\varphi_1), F(\varphi_2)) \le d_G(\varphi_1, \varphi_2)$$

Let us consider the set $\mathcal{F}_T^{all}$ of all maps F s.t. (F,T) is a GENEO.

Thm.

if Φ & Ψ are compact $\Rightarrow \mathcal{F}_T^{all}$ is compact too.

WHEN INPUT & OUTPUT IS COMPACT, THE AGENT IS COMPACT TOO

$\mathcal{F}_T^{all}$ is compact with respect to which TOPOLOGY?

$D_{GENEO}(F_1, F_2) = \sup_{\varphi \in \Phi} D_\Psi(F_1(\varphi), F_2(\varphi))$

↳ $\|F_1(\varphi) - F_2(\varphi)\|_\infty$

Thm

Let us assume that $(F_1, T) \dots (F_n, T)$ are GENEO's from (Φ, G) to (Ψ, H). Choose $(a_1 \dots a_n) \in \mathbb{R}^n$ with $\sum |a_i| \le 1 \Rightarrow (\sum a_i F_i, T)$ is a GENEO

Proof.

EQUIVALENCE

$F_\Sigma(\varphi \circ g) = \sum a_i F_i(\varphi \circ g) =$

$= \sum a_i F_i(\varphi) \circ T(g) = (\sum a_i F_i) \circ T(g) = \quad \circ T(g)$

NON-EXPANSIVE

$D_\Psi(F_\Sigma(\varphi_1), F_\Sigma(\varphi_2)) \le D_\Phi(\varphi_1, \varphi_2)$

$\quad \| $

$D_\Psi(\sum a_i F_i(\varphi_1), \sum a_i F_i(\varphi_2))$

$= \|\sum a_i F_i(\varphi_1) - \sum a_i F_i(\varphi_2)\|_\infty$

$= \|\sum a_i (F_i(\varphi_1) - F_i(\varphi_2))\|_\infty$

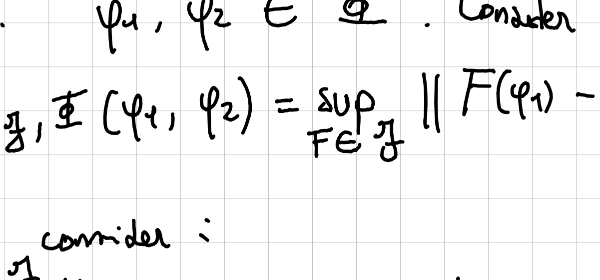
$\le \sum |a_i| \, \|F_i(\varphi_1) - F_i(\varphi_2)\|_\infty$

$\le \sum |a_i| \, \|\varphi_1 - \varphi_2\|_\infty$

$\le \|\varphi_1 - \varphi_2\|_\infty \sum |a_i|$

$\le \|\varphi_1 - \varphi_2\|_\infty$

CONVEXITY



(GENEA looking a GENEA)

Ex. $\varphi_1, \varphi_2 \in \Phi$. Consider $\mathcal{F}$ of GENEOS ← set of observers

$D_{\mathcal{F},\Phi}(\varphi_1, \varphi_2) = \sup_{F \in \mathcal{F}} \|F(\varphi_1) - F(\varphi_2)\|$

Now consider:

$D_{MATCH}^{\mathcal{F},K}(\varphi_1, \varphi_2) = \sup_{F \in \mathcal{F}} d_{match}(Dgm_K(F(\varphi_1)), Dgm_K F(\varphi_2))$

$\boxed{D_{MATCH}^{\mathcal{F},K} \le D_{\mathcal{F},\Phi}(\varphi_1, \varphi_2)}$ STABILITY

↳ IS INVARIANT ONLY IN RESPECT TO F.